Corrigé Examen EM
L2 - Physique - juin 2010

II) 1) $\Phi_1 = N_1 \iint_{S_1} \vec{B}_1 \cdot \vec{n}_1 \, dS = N_1 \mu_0 n_1 I_1 \iint_{S_1} \vec{e}_z \cdot \vec{e}_z \, d\rho \, \rho \, d\varphi$

$$\Phi_1 = N_1 \mu_0 \frac{N_1}{\ell_1} I_1 \pi R_1^2 = L_1 I_1$$

$$L_1 = \frac{N_1^2}{\ell_1} \mu_0 \pi R_1^2 = n_1^2 \ell_1 \mu_0 \pi R_1^2 \simeq \mu_0 n_1 \ell_1 \pi R^2$$

2) $\Phi_{\vec{B}_1 \to C_2} = \frac{N_2}{\ell_2} \times z \iint_{S_2} \vec{B}_1 \cdot \vec{n}_2 \, dS = n_2 n_1 \mu_0 z I_1 \pi R^2$

$$= M I_1 \implies M = n_1 n_2 \mu_0 z \pi R^2$$

3) $K = \frac{n_1 n_2 \mu_0 \pi R^2 z}{\sqrt{L_1 L_2}} = \frac{n_1 n_2 \mu_0 \pi R^2 z}{\left(\mu_0 n_1^2 \ell_1 \pi R^2 \mu_0 n_2^2 \ell_2 \pi R^2\right)^{1/2}}$

$$K = \frac{z}{\sqrt{\ell_1 \ell_2}} \quad \text{(sans dimension)}$$

$$\begin{cases} z = 0 \longrightarrow K = 0 \\ z = \ell_1 \longrightarrow K = \sqrt{\dfrac{\ell_1}{\ell_2}} \end{cases} \qquad 0 \leq K \leq \sqrt{\frac{\ell_1}{\ell_2}}$$

4) $\vec{B}_1 = \mu_0 n_1 I_1 \vec{e}_z \qquad \vec{B}_2 = \mu_0 n_2 I_2 \vec{e}_z$

$$\vec{B}_3 = \vec{B}_1 + \vec{B}_2 = \mu_0 (n_1 I_1 + n_2 I_2) \vec{e}_z$$

$$e_m = \frac{\vec{B}_1^2}{2\mu_0} + \frac{\vec{B}_2^2}{2\mu_0} + \frac{1}{2\mu_0} \vec{B}_3^2$$

(2)

$$\mathcal{E}_m = \frac{\mu_0 n_1^2 I_1^2}{2\mu_0}\iiint \rho\, d\rho\, d\varphi\, dz = \frac{\mu_0 n_1^2 I_1^2}{2}\pi R^2 \int_z^{\ell_1} dz' +$$

$$+ \frac{\mu_0^2 n_2^2 I_2^2}{2\mu_0}\iiint \rho\, d\rho\, d\varphi\, dz = \frac{\mu_0 n_2^2 I_2^2 \pi R^2}{2}\int_{-\ell_2+z}^{0} dz' +$$

$$+ \frac{\mu_0^2 (n_1 I_1 + n_2 I_2)^2}{2\mu_0}\iiint \rho\, d\rho\, d\varphi\, dz = \frac{\mu_0 (n_1 I_1 + n_2 I_2)^2}{2}\pi R^2 \int_0^z dz'$$

$$\mathcal{E}_m = \frac{\mu_0 n_1^2 I_1^2 \pi R^2}{2}(\ell_1 - z) + \frac{\mu_0 n_2^2 I_2^2 \pi R^2}{2}(\ell_2 - z)$$

$$+ \frac{\mu_0 n_1^2 I_1^2 \pi R^2 z}{2} + \mu_0 n_1 n_2 I_1 I_2 \pi R^2 z$$

$$+ \frac{\mu_0 n_2^2 I_2^2 \pi R^2 z}{2}$$

$$\mathcal{E}_m = \frac{\mu_0 n_1^2 I_1^2 \pi R^2 \ell_1}{2} + \frac{\mu_0 n_2^2 I_2^2 \pi R^2 \ell_2}{2}$$

$$+ \mu_0 n_1 n_2 I_1 I_2 \pi R^2 z$$

$$\mathcal{E}_m = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2$$

$$\Longrightarrow \boxed{M = \mu_0 n_1 n_2 \pi R^2 z}$$

III

$$\vec{E} = \begin{cases} E_x = 0 \\ E_y = E_0 \cos\left(\frac{\pi x}{2a}\right) e^{ikz} e^{-i\omega t} \\ E_z = 0 \end{cases}$$

$\vec{E}$ chp électrique transverse $\vec{E} \perp \vec{k}$

1) $\vec{\text{Rot}}\, \underline{\vec{E}} = -\frac{\partial \underline{B}}{\partial t} = i\omega \underline{B}$

$$\begin{cases} \left(\frac{\partial \underline{E}_z}{\partial y} - \frac{\partial \underline{E}_y}{\partial z}\right) = i\omega \underline{B}_x = -\frac{\partial \underline{E}_y}{\partial z} \\ \left(\frac{\partial \underline{E}_x}{\partial z} - \frac{\partial \underline{E}_z}{\partial x}\right) = i\omega \underline{B}_y = 0 \\ \left(\frac{\partial \underline{E}_y}{\partial x} - \frac{\partial \underline{E}_x}{\partial y}\right) = i\omega \underline{B}_z \end{cases}$$

$$\begin{cases} i\omega \underline{B}_x = -\frac{\partial \underline{E}_y}{\partial z} = -ik E_0 \cos\left(\frac{\pi x}{2a}\right) e^{ikz} e^{-i\omega t} \\ i\omega \underline{B}_z = \frac{\partial \underline{E}_y}{\partial x} = -E_0 \left(\frac{\pi}{2a}\right) \sin\left(\frac{\pi x}{2a}\right) e^{ikz} e^{-i\omega t} \\ i\omega \underline{B}_y = 0 \end{cases}$$

$$\begin{cases} \underline{B}_x = -\frac{k}{\omega} E_0 \cos\left(\frac{\pi x}{2a}\right) e^{i(kz-\omega t)} \\ \underline{B}_y = 0 \\ \underline{B}_z = i \frac{E_0}{\omega}\left(\frac{\pi}{2a}\right) \sin\left(\frac{\pi x}{2a}\right) e^{i(kz-\omega t)} \end{cases}$$

$\underline{B}_z \neq 0$ champ magnétique non transverse

2) $$\operatorname{div} \vec{\underline{B}} = \left(\frac{\partial \underline{B}_x}{\partial x}\right) + \left(\frac{\partial \underline{B}_y}{\partial y}\right) + \left(\frac{\partial \underline{B}_z}{\partial z}\right)$$

$$= \frac{k}{\omega} E_0 \left(\frac{\pi}{2a}\right) \sin\left(\frac{\pi x}{2a}\right) e^{i(kz - \omega t)}$$

$$+ i \frac{E_0}{\omega} \left(\frac{\pi}{2a}\right) \sin\left(\frac{\pi x}{2a}\right) e^{i(kz - \omega t)} ik$$

$$\operatorname{div} \vec{B} = \left\{ \frac{k E_0}{\omega} \left(\frac{\pi}{2a}\right) \sin\left(\frac{\pi x}{2a}\right) - \frac{k E_0}{\omega} \left(\frac{\pi}{2a}\right) \sin\left(\frac{\pi x}{2a}\right) \right\} e^{i(kz - \omega t)} = 0$$

3) $\vec{\underline{B}}(x = -a) \to \underline{B}_x(-a) = 0$

$\to \underline{B}_z(-a) = \frac{i E_0}{\omega} \left(\frac{\pi}{2a}\right) e^{i(kz - \omega t)}$

$\vec{\underline{B}}(x = +a) \to \underline{B}_x(-a) = 0$

$\to \underline{B}_z(-a) = -i \frac{E_0}{\omega} \left(\frac{\pi}{2a}\right) e^{i(kz - \omega t)}$

$$\boxed{\underline{B}_x (x = \pm a) = 0}$$

$$\boxed{\underline{B}_z (x = +a) = - \underline{B}_z (x = -a)}$$

4) $$\widehat{\operatorname{Rot}} \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

$$\operatorname{Rot} \vec{\underline{B}} = - \frac{i\omega}{c^2} \vec{\underline{E}}$$

$$\left(\frac{\partial \underline{B}_z}{\partial y} - \frac{\partial \underline{B}_y}{\partial z}\right) = - \frac{i\omega}{c^2} \underline{E}_x = 0$$

$$\left(\frac{\partial \underline{B}_x}{\partial z} - \frac{\partial \underline{B}_z}{\partial x}\right) = - \frac{i\omega}{c^2} \underline{E}_y \quad (1)$$

$$\left(\frac{\partial \underline{B}_y}{\partial x} - \frac{\partial \underline{B}_x}{\partial y}\right) = - \frac{i\omega}{c^2} \underline{E}_z = 0$$

$$① \Longleftrightarrow i\frac{k^2}{\omega}E_0\cos\left(\frac{\pi x}{2a}\right)e^{i(kz-\omega t)}$$

$$-i\frac{E_0}{\omega}\left(\frac{\pi}{2a}\right)^2\cos\left(\frac{\pi x}{2a}\right)e^{i(kz-\omega t)}$$

$$① \Longleftrightarrow -i\frac{E_0}{\omega}\cos\left(\frac{\pi x}{2a}\right)e^{i(kz-\omega t)}\left\{k^2+\left(\frac{\pi}{2a}\right)^2\right\} =$$

$$-i\frac{\omega}{c^2}E_0\cos\left(\frac{\pi x}{2a}\right)e^{i(kz-\omega t)}$$

$$\Longrightarrow \left\{k^2+\left(\frac{\pi}{2a}\right)^2\right\} = \frac{\omega^2}{c^2}$$

$$\boxed{k^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{2a}\right)^2}$$

$$\frac{\omega}{k} = \frac{c}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}} = \frac{c\,\omega}{\sqrt{\omega^2-\omega_c}}$$

$$2k\,dk = \frac{1}{c^2}2\omega\,d\omega \qquad \frac{d\omega}{dk}\,\frac{\omega}{k} = c^2$$

$$v_g = \frac{d\omega}{dk} \qquad v_\varphi = \frac{\omega}{k} = \frac{c}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}$$

5) $\omega < \omega_c \quad k^2 < 0 \Longleftrightarrow k = ik'$

$$E_x = E_0\cos\left(\frac{\pi x}{2a}\right)e^{-i\omega t}e^{-k'z}$$

onde évanescente suivant z'

6) $\vec{R} = \dfrac{\vec{E} \wedge \vec{B}}{\mu_0} = \dfrac{1}{\mu_0} \begin{vmatrix} 0 \\ \underline{E}_y \\ 0 \end{vmatrix} \wedge \begin{vmatrix} \underline{B}_x \\ 0 \\ \underline{B}_z \end{vmatrix}$

$$\vec{R} = \frac{1}{\mu_0} \begin{vmatrix} \underline{E}_y \ \underline{B}_z \\ 0 \\ -\underline{B}_x \ \underline{E}_y \end{vmatrix}$$

$$\underline{R}_x = \frac{i}{\mu_0} \frac{E_0^2}{\omega} \cos\left(\frac{\pi x}{2a}\right) e^{i(kz-\omega t)} \sin\left(\frac{\pi x}{2a}\right)\left(\frac{\pi}{2a}\right) e^{i(kz-\omega t)}$$

$$\underline{R}_y = 0$$

$$\underline{R}_z = \frac{E_0^2}{\mu_0 \omega} k \cos^2\left(\frac{\pi x}{2a}\right) e^{i(kz-\omega t)} e^{i(kz-\omega t)}$$

$$R_x = i \frac{E_0^2}{\mu_0 \omega}\left(\frac{\pi}{2a}\right) \cos\left(\frac{\pi x}{2a}\right) \sin\left(\frac{\pi x}{2a}\right) \cos^2(kz-\omega t)$$

$$R_z = \frac{E_0^2 k}{\mu_0 \omega} \cos^2\left(\frac{\pi x}{2a}\right) \cos^2(kz-\omega t)$$

$$\langle \vec{R} \rangle_t = \frac{E_0^2 k}{2\mu_0 \omega} \cos^2\left(\frac{\pi x}{2a}\right) \vec{e_z}$$

$$= \frac{1}{2} \frac{E_0^2}{c\mu_0} \cos^2\left(\frac{\pi x}{2a}\right) \vec{e_z}$$

7) $\langle P \rangle_t = \iint_S \langle \vec{R} \rangle_t \vec{n}\, dS = \dfrac{1}{2} \dfrac{E_0^2 b}{c\mu_0} \displaystyle\int_{-a}^{a} \cos^2\left(\frac{\pi x}{2a}\right) dx$

$$\langle P \rangle_t = \frac{E_0^2}{2} \frac{ab}{c\mu_0}$$

8) $\langle e_m \rangle_t = \left\langle \dfrac{\varepsilon_0 E_x^2}{2} \right\rangle_t + \left\langle \dfrac{B_x^2 + B_z^2}{2\mu_0} \right\rangle_t$

$$\langle e_m \rangle_t = \frac{\varepsilon_0 E_0^2}{4} \cos^2\left(\frac{\pi x}{2a}\right) + \frac{E_0^2}{4 \mu_0 \omega^2} \left[k^2 \cos^2\left(\frac{\pi x}{2a}\right) + \left(\frac{\pi}{2a}\right)^2 \sin^2 \frac{\pi x}{2a} \right]$$

$$\langle\langle e_m \rangle_t\rangle_S = \frac{\varepsilon_0 E_0^2}{8} ab + \frac{E_0^2 ab}{8 \mu_0 \omega^2} \left[k^2 + \left(\frac{\pi}{2a}\right)^2 \right]$$

$$= \frac{\varepsilon_0 E_0^2}{8} ab \left[1 + \frac{1}{\mu_0} \left(\frac{\omega}{c}\right)^2 \times \frac{1}{\omega^2} \right]$$

$$= \frac{E_0^2}{8} ab \left[\varepsilon_0 + \frac{1}{\mu_0 c^2} \right]$$

$$= \frac{E_0^2}{8} ab \left[\varepsilon_0 + \frac{\varepsilon_0 \cancel{\mu_0}}{\cancel{\mu_0}} \right]$$

$$\boxed{\langle\langle e_m \rangle_t\rangle_S = \varepsilon_0 \frac{E_0^2 ab}{4}}$$

g) $$\frac{\langle P \rangle_t}{\langle\langle e_m \rangle_t\rangle_S} = \frac{1}{c \mu_0 \varepsilon_0} = c$$